Infinitely many conservation laws in self-dual Yang-Mills theory

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ABSTRACT: Using a nonlocal field transformation for the gauge field known as Cho-Faddeev-Niemi-Shabanov decomposition as well as ideas taken from generalized integrability, we derive a new family of infinitely many conserved currents in the self-dual sector of SU(2) Yang-Mills theory. These currents may be related to the area preserving diffeomorphisms on the reduced target space. The calculations are performed in a completely covariant manner and, therefore, can be applied to the self-dual equations in any space-time dimension with arbitrary signature.

Contents

1. Introduction

A powerful tool in the theory of topological solitons is the derivation of lower bounds for the energy (or Euclidean action) in terms of topological charges. Together with these bounds, in some cases one may derive first order equations (so-called Bogomolny equations) such that any field configuration obeying these Bogomolny equations automatically saturates the topological lower bound and is a true minimizer of the energy (or Euclidean action) functional. Obviously, any field configuration obeying the Bogomolny equations automatically obeys the original second order Euler-Lagrange equations, whereas the converse is not true in general.

In addition to providing true minimizers of the energy functional, these Bogomolny equations, due to their more restrictive nature, tend to enhance the number of symmetries and conservation laws. Sometimes, there exist infinitely many symmetries and infinitely many conservation laws for the Bogomolny equations. Further, the Bogomolny equations are usually not of the Euler-Lagrange type, therefore for those symmetries which are not symmetries of the original second order system, the issue of conservation laws has to be investigated separately, that is, Noether's theorem does not apply. A theory where this happens is, for instance, the CP(1) model in 2+1 dimensions. For this theory both the

infinitely many symmetries and the infinitely many conservation laws of the Bogomolny sector have been calculated, e.g., in [1], and, indeed, they turn out to be different. Similar investigations for gauge theories have been performed recently. In the case of the Abelian Higgs model, an equivalent pattern has been found, i.e., there are infinitely many conserved currents in the Bogomolny sector, and Noether's theorem does not apply, see [2]. A slightly different scenario is realized in the Abelian projection of Yang-Mills dilaton theory. There, too, exists a Bogomolny sector, but this theory has infinitely many symmetries already on the level of the Lagrangian, therefore the symmetries and conservation laws are related by Noether's theorem, see [3].

In the case of SU(2) Yang-Mills theory, the solutions which minimize the Euclidean action functional are known as instantons, and the Bogomolny type first order equations are the self-duality equations [4]–[7]. The symmetries of the self-dual sector of SU(2) Yang-Mills theory have been studied by various authors [8]–[17]. The result is that the system posesses infinitely many symmetries and that almost all of them are nonlocal when expressed in terms of the original fields. A recent review of this issue can be found in [18], to which we refer the reader for further information and additional references. Conservation laws of self-dual Yang-Mills theory related to the non-local symmetries mentioned above have been studied, e.g., in [12]–[17].

A slightly different, more geometric approach to the self-dual Yang-Mills (SDYM) equations focusing directly on their integrability has been initiated by R. Ward [19]. In that approach twistor methods are employed, and the use of twistor methods in the investigation of the SDYM and their conservation laws has played an important role subsequently (for some recent results, see [20]–[23]).

Another approach to integrability and conservation laws has been proposed in [24], where a generalized zero curvature representation suitable for higher-dimensional field theories was developed, analogously to the zero curvature representation of Zakharov and Shabat, which provides integrable field theories in 1+1 dimensions. Among other results, it was demonstrated in that paper that the SDYM permit a generalized zero curvature representation. But still only finitely many conservation laws have been provided for self-dual Yang-Mills theory in ref. [24]. It is the purpose of the present paper to further develop the issue of integrability and conservation laws of the self-dual sector of SU(2) Yang-Mills theory along these lines. We will find another set of infinitely many conservation laws by explicit construction. The corresponding conserved currents are nonlocal in terms of the original Yang-Mills field, but they will be local in terms of a well-known nonlocal field redefinition which we shall use in the sequel. In contrast to the conserved currents found previously, the ones we shall present below are given by manifestly Lorentz covariant expressions and may, therefore, easily be generalized to different space time metrics and dimensions. Given the relevance of self-dual Yang-Mills theories both for strong interaction physics and in a more mathematical context, we believe that the discovery of these additional conserved currents is of some interest.

We want to remark that in a recent paper devoted to similar problems [25], an investigation of integrability in the sector of Z_N string solutions of Yang-Mills theory has been performed. Z_N string solutions are effectively lower dimensional solutions, but, nev-

ertheless, they also belong to the self-dual sector. Further, the integrability of self-dual Yang-Mills theories on certain four-dimensional product manifolds has been used in [26, 27] to demonstrate the integrability of abelian and nonabelian Higgs models on general Riemannian surfaces.

Our paper is organized as follows. In section 2 we present a brief overview of some known results on the self-dual Yang-Mills (SDYM) equations. Specifically, we present infinitely many nonlocal conserved currents as constructed by Prasad et al and by Papachristou. This overview shall serve later on to relate our own findings to these already known results. In section 3 we recapitulate how the self-dual sector of SU(2) Yang-Mills theory may be recast into the form of the generalized zero curvature representation. In section 4 we introduce the Cho-Faddeev-Niemi-Shabanov (CFNS) decomposition of the gauge field and re-express the self-dual equations using this decomposition. Next, we write down the currents in terms of the decomposition fields and prove that they are conserved. Section 5 contains our conclusions. In the appendix we display the canonical four momenta and field equations which we need in the main text.

2. Some known facts about the SDYM

2.1 J formulation of the SDYM

The self-dual sector of SU(2) Yang-Mills theory in Euclidean space-time is constituted by gauge fields A_μ^a satisfying the following equations

$$F_{\mu\nu}^a = {}^* F_{\mu\nu}^a, \tag{2.1}$$

where

$$F_{\mu\nu}^a \equiv \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + \epsilon^{abc} A_\mu^b A_\nu^c, \quad {}^* F_{\mu\nu}^a \equiv \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{a\rho\sigma} \tag{2.2}$$

It is convenient to rewrite them as

$$F_{yz}^a = 0, \quad F_{\bar{y}\bar{z}}^a = 0, \quad F_{y\bar{y}}^a + F_{z\bar{z}}^a = 0, \tag{2.3}$$

where the new independent variables are defined as

$$y = \frac{1}{\sqrt{2}}(x_1 + ix_2), \quad \bar{y} = \frac{1}{\sqrt{2}}(x_1 - ix_2), \quad z = \frac{1}{\sqrt{2}}(x_3 - ix_4), \quad \bar{z} = \frac{1}{\sqrt{2}}(x_3 + ix_4).$$

Defining the self-dual gauge fields as

$$A_y^a = g_1^{-1} \partial_y g_1, \quad A_z^a = g_1^{-1} \partial_z g_1, \quad A_{\bar{y}}^a = g_2^{-1} \partial_{\bar{y}} g_2, \quad A_{\bar{z}}^a = g_2^{-1} \partial_{\bar{z}} g_2 \tag{2.4}$$

we identically fulfill the first two equations in (2.3). Here, g_1, g_2 are arbitrary group elements in SU(2). Then the third expression leads to a nontrivial equation giving an equivalent formulation of the self-dual equations

$$F[J] \equiv \partial_{\bar{y}} \left(J^{-1} \partial_y J \right) + \partial_{\bar{z}} \left(J^{-1} \partial_z J \right) = 0, \tag{2.5}$$

where

$$J = g_1 g_2^{-1}. \tag{2.6}$$

In other words, solutions of the self-dual sector are defined by eq. (2.5).

2.2 Linear system, Bäcklund transformation and hidden symmetry

There is a formulation of the SDYM equations in terms of a linear system [8]–[10]. Namely, consider an auxiliary matrix field ψ defined by the following set of equations

$$\partial_{\bar{z}}\psi = \lambda\left(\partial_y\psi + J^{-1}J_y\psi\right), \quad -\partial_{\bar{y}}\psi = \lambda\left(\partial_z\psi + J^{-1}J_z\psi\right). \tag{2.7}$$

In fact, this is just the Lax pair formulation. The SDYM equations (2.5) are derived as a consistency (integrability) condition $\psi_{\bar{z}\bar{y}} = \psi_{\bar{y}\bar{z}}$.

Further, it is possible to find the related Bäcklund transformation (BT) [8]. It is given by

$$J'^{-1}J'_y - J^{-1}J_y = \lambda(J'^{-1}J)_{\bar{z}}, \quad J'^{-1}J'_z - J^{-1}J_z = \lambda(J'^{-1}J)_{\bar{y}}. \tag{2.8}$$

If J is a solution of SDYM then J' is a new solution of SDYM. Further, this BT is an infinitesimal BT i.e., a new solution generated by the BT may be found performing an infinitesimal transformation which leaves the SDYM equation invariant. The pertinent transformation reads

$$J^{-1}\delta J = -\psi T_a \psi^{-1}\alpha^a \tag{2.9}$$

where T_a is a basis of the Lie algebra for the gauge fields and α^a are infinitesimal parameters. Indeed, if we assume that $J' = J + \delta J$ then we get the BT.

This symmetry transformation gives the following commutator

$$[\delta_\alpha, \delta_\beta]J = \alpha^a\beta^b C_{ab}^c \frac{d}{d\lambda}(\lambda\delta_c J), \tag{2.10}$$

where C_{ab}^c are the structure constants of the Lie algebra. If we expand $\psi = \sum_{n=0}^{\infty}\lambda^n\psi^{(n)}$ then we get the Kac-Moody algebra

$$[\delta_\alpha^{(m)}, \delta_\beta^{(n)}]J = \alpha^a\beta^b C_{ab}^c \lambda\delta_c^{(m+n)}J, \tag{2.11}$$

where $\delta^{(n)}$ is defined as $J^{-1}\delta J = \sum_{n=0}^{\infty}\lambda^n J^{-1}\delta^{(n)}J$ and $\delta^{(n)}J = -J\sum_{n=0}^{\infty}\psi^{(n)}T\psi^{(m-n)}$. In this way, one may explain the hidden (infinite) symmetries observed by L. Dolan [11]. It is precisely the symmetry transformation (2.9) mentioned above.

2.3 Nonlocal conservation laws

The SDYM in J formulation is an Euler-Lagrange system, and the Noether theorem applies. Indeed, eq. (2.5) may be easily derived from the Euclidean action

$$S = \int d^4x \text{Tr}[(J^{-1}\partial^\mu J)(J^{-1}\partial_\mu J)]. \tag{2.12}$$

Therefore, the derivation of an infinite set of symmetries indicates that there should exist infinitely many conserved quantities, as is expected in any case for an integrable system. In fact, several families of nonlocal conserved currents have been found. All these constructions use in an essential way the J-formulation of the self-dual sector and therefore are unique for 4-dimensional Euclidean space-time. Moreover, manifest Lorentz covariance is lost since we introduced the complex variables y, z. On the other hand, this formulation

of the self-dual equations possesses the advantage that equation (2.5) has the form of a conservation law.

The first set of conserved currents was discovered by Prasad et al [12, 13]. The construction reads as follows. Let us rewrite the SDYM equation as

$$(v_y^{(n)})_{\bar{y}} + (v_z^{(n)})_{\bar{z}} = 0 \quad \text{and} \quad v_y^{(1)} = J^{-1}J_y, \; v_z^{(1)} = J^{-1}J_z, \tag{2.13}$$

where $v_y^{(n)}, v_y^{(n)}, n = 1, 2, 3 \ldots$ are higher conserved currents, which can be constructed by induction (iteratively). One has to define a set of potentials $X^{(n)}$

$$v_y^{(n)} = \partial_{\bar{z}}X^{(n)}, \quad v_z^{(n)} = -\partial_{\bar{y}}X^{(n)}, \quad X^{(0)} = I. \tag{2.14}$$

Then, if the n-th current has been found, the next one is given by the formula

$$v_y^{(n+1)} = (\partial_y + J^{-1}J_y)X^{(n)}, \quad v_z^{(n+1)} = (\partial_z + J^{-1}J_z)X^{(n)}. \tag{2.15}$$

A different family of nonlocal conservation laws, nontrivially related to Prasad's ones, was presented by Papachristou [14]. The basic idea was to reformulate the SDYM equation using the potential symmetries. At the beginning we have a SDYM field J obeying $F[J] = 0$ and introduce a potential X (similarly as in Prasad's work)

$$J^{-1}J_y \equiv X_{\bar{z}}, \quad J^{-1}J_z \equiv -X_{\bar{y}}. \tag{2.16}$$

The consistency (integrability) condition $(X_{\bar{z}})_{\bar{y}} = (X_{\bar{y}})_{\bar{z}}$ gives $F[J] = 0$, whereas the condition $(J_y)_z = (J_z)_y$ leads to the potential SDYM equation (PSDYM)

$$G[X] \equiv X_{y\bar{y}} + X_{z\bar{z}} - [X_{\bar{y}}, X_{\bar{z}}] = 0. \tag{2.17}$$

The point is that this expression may also be written as a conservation law

$$\partial_{\bar{y}}\left(X_y - \frac{1}{2}[X, X_{\bar{z}}]\right) + \partial_{\bar{z}}\left(X_z + \frac{1}{2}[X, X_{\bar{y}}]\right) = 0. \tag{2.18}$$

Therefore, we arrive at a new current. This procedure may be repeated. We introduce a new potential Y to the last formula

$$X_y - \frac{1}{2}[X, X_{\bar{z}}] = Y_{\bar{z}}, \quad X_z + \frac{1}{2}[X, X_{\bar{y}}] = Y_{\bar{y}} \tag{2.19}$$

and consider the consistency condition $(X_y)_z = (X_z)_y$. As a result we derive a new PSDYM equation which has the form of a conservation law, as well. One may continue with this procedure and, at least in principle, derive an infinite set of conserved quantities. There is some similarity between the two sets of currents, however, the relation between them is non-trivial [14].

The importance of the PSDYM equation originates in the observation that there is a one-to-one correspondence between symmetries of the SDYM and PSDYM, as it was formulated in the theorem by Papachristou [15]

$$\delta X = \alpha\Phi \text{ is a symmetry of } G[X] \Leftrightarrow \delta J = \alpha J\Phi \text{ is a symmetry of } F[J], \tag{2.20}$$

where $X \to X' = X + \alpha\Phi$ is a transformation which leaves the PSDYM invariant: $G[X'] = 0$ if $G[X] = 0$, or in other words

$$\delta G \equiv H(\Phi) = \Phi_{y\bar{y}} + \Phi_{z\bar{z}} + [X_{\bar{z}}, \Phi_{\bar{y}}] - [X_{\bar{y}}, \Phi_{\bar{z}}] = 0. \tag{2.21}$$

The next step is to find a Bäcklund transformation generating the symmetries of the PSDYM [15]

$$\lambda\Phi'_{\bar{z}} = \Phi_y + [X_{\bar{z}}, \Phi], \quad \lambda\Phi'_{\bar{y}} = -\Phi_z + [X_{\bar{y}}, \Phi], \tag{2.22}$$

provided X is any given solution of the PSDYM equation, for example (2.16). Then starting with any local symmetry of the PSDYM $\Phi^{(0)}$ (or the SDYM as they are in one-to-one correspondence) one is able to construct an infinite tower of symmetries $\{\Phi^{(n)}\}_{n=0}^{\infty}$. Moreover, as the Bäcklund transformation (2.22) immediately provides a conservation law we get an infinite series of conserved quantities, each based on a particular local symmetry of the SDYM equations.

The extensive analysis of such families of conserved quantities has been performed by Papachristou [16]. He introduced a recursion operator $\hat{R}$ which transforms one symmetry of the PSDYM equation into another one and is given by a formal integration of the Bäcklund transformation (2.22)

$$\hat{R} \equiv \partial_{\bar{z}}^{-1}(\partial_y + [X_{\bar{z}},]). \tag{2.23}$$

To be precise, he constructed an infinite set of Lie derivatives $\Delta^{(n)}X = \Phi^{(n)}$, where $\Phi^{(n)}$ is a symmetry of the PSDYM equation as

$$\Delta_k^{(n)}X = R^{(n)}L_kX. \tag{2.24}$$

Here L_k is a symmetry operator for the PSDYM equation corresponding to a given *local* symmetry. The results may be summarized as follow.

For internal symmetries $\Phi \equiv \Delta_kX \equiv L_kX = [X, T_k]$, where T_k is a basis for the $su(2)$ Lie algebra of the gauge fields, we get that the infinite set of transformations

$$\Delta_k^{(n)}X = R^{(n)}L_kX = R^{(n)}[X, T_k] \tag{2.25}$$

obeys the Kac-Moody algebra

$$[\Delta_i^{(m)}, \Delta_j^{(n)}]X = C_{ij}^k \Delta_k^{(m+n)}X. \tag{2.26}$$

Once again, it is exactly the hidden symmetry of SDYM found by L. Dolan.

Finally let us discuss the nine local (point) symmetries of the SDYM

$$
\begin{aligned}
&L_1 = \partial_y, &&L_2 = \partial_z, \\
&L_3 = z\partial_y - \bar{y}\partial_{\bar{z}}, &&L_4 = y\partial_z - \bar{z}\partial_{\bar{y}}, \\
&L_5 = y\partial_y - z\partial_z - \bar{y}\partial_y + \bar{z}\partial_z, &&L_6 = 1 + y\partial_y + z\partial_z, \\
&L_7 = 1 - \bar{y}\partial_y - \bar{z}\partial_z, &&L_8 = yL_6 + \bar{z}(y\partial_{\bar{z}} - z\partial_{\bar{y}}), \\
&L_9 = zL_6 + \bar{y}(z\partial_{\bar{y}} - y\partial_{\bar{z}})
\end{aligned}
\tag{2.27}
$$

The subset $\{L_1 \ldots L_5\}$ provides an infinite set of transformations

$$\Delta_k^{(n)} X = R^{(n)} L_k X, \qquad k = 1 \ldots 5 \qquad (2.28)$$

satisfying also a Kac-Moody algebra. Additionally, L_6 and L_7 give two sets of infinitely many transformations

$$\Delta^{(n)} X = R^{(n)} L X, \quad L = L_6 \text{ or } L_7 \qquad (2.29)$$

leading to two copies of the Virasoro algebra. Generators L_8 and L_9 probably do not result in any algebraic structure.

Obviously, conservation laws do not have to correspond to conserved charges. This happens, e.g., if the spatial integrals of the fluxes (charge densities) do not converge. As observed by Ioannidou and Ward [17], the nonlocal currents found by Prasad [12] and Papachristou [14, 15] lead to densities which diverge after integration. To be precise, it was discussed for the chiral model in (2+1) dimension but these results should hold also for SDYM. A general argument is the following. All nonlocal conserved currents of type [12, 14 – 16] are constructed using the integral operator ∂^{-1} and, further, the instanton field is power-like localized. Thus, after a sufficient number of integrations we arrive at a divergent quantity.

3. Generalized integrability in the SDYM

Here, following [24], we very briefly describe the self-dual sector of SU(2) Yang-Mills theory in the language of generalized integrability. The basic step in this framework is the choice of a reducible Lie algebra $\tilde{\mathcal{G}} = \mathcal{G} \oplus \mathcal{H}$, where $\mathcal{G}$ is a Lie algebra and $\mathcal{H}$ is an Abelian ideal (in practice, a representation space of $\mathcal{G}$), together with a connection $\mathcal{A}_\mu \in \mathcal{G}$ and a vector field $\mathcal{B}_\mu \in \mathcal{H}$. A system possesses the generalized zero curvature representation if its equations of motion may be encoded in two conditions. Namely, the flatness of the connection

$$\partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + [\mathcal{A}_\mu, \mathcal{A}_\nu] = 0 \qquad (3.1)$$

and the covariant constancy of the vector field

$$\partial_\mu \mathcal{B}^\mu + [\mathcal{A}_\mu, \mathcal{B}^\mu] = 0. \qquad (3.2)$$

Usually, one assumes a trivial connection i.e., $A_\mu = g^{-1}\partial_\mu g$, where $g \in G$. In this case, one can easily construct conserved currents

$$\mathcal{J}_\mu = g\mathcal{B}_\mu g^{-1}.$$

We say that a system is integrable if the number of currents is infinite. As it is equal to the dimension of the Abelian ideal $\mathcal{H}$, the integrability condition is simply $\dim \mathcal{H} = \infty$.

Let us now express the self-dual equations of SU(2) YM in this manner. Again, we use the representation of the self-dual equations via the equation for the J matrix. In order to

accomplish that we introduce a flat connection $\mathcal{A}_\mu$ and a covariantly constant vector $\mathcal{B}_\mu$ taking values in an Abelian ideal in the following way

$$\mathcal{A}_\mu = J^{-1}\partial_\mu J = \mathcal{A}_\mu^r \mathcal{T}_r \tag{3.3}$$

$$\mathcal{B}_{\bar{y}} = \mathcal{A}_{\bar{y}}^r \mathcal{S}_r, \quad \mathcal{B}_{\bar{z}} = \mathcal{A}_{\bar{z}}^r \mathcal{S}_r, \quad \mathcal{B}_y = 0, \quad \mathcal{B}_z = 0, \tag{3.4}$$

where $\mathcal{T}_r$, $\mathcal{S}_r$ form a basis satisfying

$$[\mathcal{T}_r, \mathcal{T}_s] = C_{rs}^u \mathcal{T}_u, \quad [\mathcal{T}_r, \mathcal{S}_s] = C_{rs}^u \mathcal{S}_u, \quad [\mathcal{S}_r, \mathcal{S}_s] = 0.$$

Obviously, the connection is flat as it is a pure gauge configuration. Moreover the condition for the vector field i.e., $D_\mu \mathcal{B}^\mu = 0$ is equivalent to the self-dual equation (2.5). One can construct conserved currents

$$\mathcal{J}_{\bar{y}} = \mathcal{A}_{\bar{y}}^r J \mathcal{S}_r J^{-1} \quad \mathcal{J}_{\bar{z}} = \mathcal{A}_{\bar{z}}^r J \mathcal{S}_r J^{-1}, \quad \mathcal{J}_y = 0, \quad \mathcal{J}_z = 0, \tag{3.5}$$

then, the conservation laws are just the self-dual equations (2.5). More conservation laws may be derived as discussed in the previous section.

Of course, the obtained result is not surprising. A system which possesses the standard zero curvature representation admits also the generalized zero curvature formulation. However, there is a simple prescription how to construct an infinite family of additional conserved currents for a model with generalized zero curvature formulation. In general, they are spanned by the canonical momenta conjugated to the field degrees of freedom. It is important to check whether such currents can be also found for the self-dual sector of the SU(2) YM theory, and, if the answer is positive, what is their relation with the standard non-local conservation laws described before.

4. New conserved currents in the SDYM

4.1 Cho-Faddeev-Niemi-Shabanov decomposition

In order to derive such conserved quantities in an exact form we perform a nonlocal change of variables known as the Cho-Faddeev-Niemi-Shabanov decomposition [28]–[33]. The decomposition

$$\vec{A}_\mu = C_\mu \vec{n} + \partial_\mu \vec{n} \times \vec{n} + \vec{W}_\mu \tag{4.1}$$

relates the original SU(2) non-Abelian gauge field with three fields: a three component unit vector field $\vec{n}$ pointing into the color direction, an Abelian gauge potential C_μ and a color vector field W_μ^a which is perpendicular to $\vec{n}$. The fields are not independent. In fact, as we want to keep the correct gauge transformation properties

$$\delta n^a = \epsilon^{abc} n^b \alpha^c, \quad \delta W_\mu^a = \epsilon^{abc} W_\mu^b \alpha^c, \quad \delta C_\mu = n^a \alpha_\mu^a \tag{4.2}$$

under the primary gauge transformation

$$\delta A_\mu^a = (D_\mu \alpha)^a = \alpha_\mu^a + \epsilon^{abc} A_\mu^b \alpha^c \tag{4.3}$$

one has to impose the constraint ($n_\mu^b \equiv \partial_\mu n^b$ etc.)

$$\partial^\mu W_\mu^a + C^\mu \epsilon^{abc} n^b W_\mu^c + n^a W^{b\mu} n_\mu^b = 0. \tag{4.4}$$

In the subsequent analysis we assume a particular form for the valence field W_μ^a. It is equivalent to a partial gauge fixing where one leaves a residual local $\mathrm{U}(1)$ gauge symmetry. Namely,

$$W_\mu^a = \rho n_\mu^a + \sigma \epsilon^{abc} n_\mu^b n^c, \tag{4.5}$$

where ρ, σ are real scalars. For reasons of convenience we combine them into a complex scalar $v = \rho + i\sigma$. Then the Lagrange density takes the form ($u_\mu \equiv \partial_\mu u$ etc.)

$$
\begin{aligned}
L = {}& F_{\mu\nu}^2 - 2(1 - |v|^2) H_{\mu\nu} + (1 - |v|^2)^2 H_{\mu\nu}^2 \\
& + \frac{8}{(1 + |u|^2)^2} \left[(u_\mu \bar{u}^\mu)(D^\nu v \overline{D_\nu v}) - (D_\mu v \bar{u}^\mu)(\overline{D_\nu v} u^\nu) \right],
\end{aligned} \tag{4.6}
$$

where

$$H_{\mu\nu} = \vec{n} \cdot [\vec{n}_\mu \times \vec{n}_\nu] = \frac{-2i}{(1 + |u|^2)^2}(u_\mu \bar{u}_\nu - u_\nu \bar{u}_\mu), \quad H_{\mu\nu}^2 = \frac{8}{(1 + |u|^2)^4}[(u_\mu \bar{u}^\mu)^2 - u_\mu^2 \bar{u}_\nu^2] \tag{4.7}$$

and the covariant derivatives read $D_\mu v = v_\mu - ieC_\mu v$, $\overline{D_\mu v} = \bar{v}_\mu + ieC_\mu \bar{v}$ and we expressed the unit vector field by means of the stereographic projection

$$\vec{n} = \frac{1}{1 + |u|^2} \left(u + \bar{u}, -i(u - \bar{u}), |u|^2 - 1 \right).$$

Further,

$$F_{\mu\nu} \equiv \partial_\mu C_\nu - \partial_\nu C_\mu$$

is the Abelian field strength tensor corresponding to the Abelian gauge field C_μ. Notice that only the complex field v couples to the gauge field via the covariant derivative.

4.2 Self-dual equations

Now, we apply the CFNS decomposition to the self-dual equations. As we know the full field strength tensor reads

$$
\begin{aligned}
\vec{F}_{\mu\nu} = {}& \left[F_{\mu\nu} - (1 - |v|^2) H_{\mu\nu} \right] \vec{n} + \frac{1}{2} \left[(D_\mu v + \overline{D_\mu v}) \vec{n}_\nu - (D_\nu v + \overline{D_\nu v}) \vec{n}_\mu \right] + \\
& + \frac{1}{2i} \left[(D_\mu v - \overline{D_\mu v}) \vec{n}_\nu \times \vec{n} - (D_\nu v - \overline{D_\nu v}) \vec{n}_\mu \times \vec{n} \right].
\end{aligned} \tag{4.8}
$$

Therefore, using the self-dual equations (2.1) we get two expressions, one parallel and one perpendicular to the color vector $\vec{n}$

$$\frac{1}{2} \epsilon_{\mu\nu\rho\sigma}[F^{\rho\sigma} - (1 - |v|^2) H^{\rho\sigma}] = F_{\mu\nu} - (1 - |v|^2) H_{\mu\nu} \tag{4.9}$$

and

$$\frac{1}{2}\epsilon_{\mu\nu}{}^{\rho\sigma}\Big[\big[(D_\rho v+\overline{D_\rho v})\vec{n}_\sigma-(D_\sigma v+\overline{D_\sigma v})\vec{n}_\rho\big] \tag{4.10}$$
$$-i\big[(D_\rho v-\overline{D_\rho v})\vec{n}_\sigma\times\vec{n}-(D_\sigma v-\overline{D_\sigma v})\vec{n}_\rho\times\vec{n}\big]\Big]=$$
$$\big[(D_\mu v+\overline{D_\mu v})\vec{n}_\nu-(D_\nu v+\overline{D_\nu v})\vec{n}_\mu\big]$$
$$+i\big[(D_\mu v-\overline{D_\mu v})\vec{n}_\nu\times\vec{n}-(D_\nu v-\overline{D_\nu v})\vec{n}_\mu\times\vec{n}\big].$$

For later convenience we now want to derive some constraints which result from these two sets of equations. On the one hand, after projection on $\vec{n}_\mu$, eq. (4.10) gives

$$(D_\mu v+\overline{D_\mu v})\vec{n}^\mu\cdot\vec{n}_\nu-(D_\nu v+\overline{D_\nu v})\,\vec{n}_\mu^2-i(D^\mu v-\overline{D^\mu v})H_{\mu\nu}=-i\epsilon^{\mu\nu\lambda\omega}(D_\lambda v-\overline{D_\lambda v})H_{\mu\omega}. \tag{4.11}$$

On the other hand, if we multiply (4.10) by $\times\vec{n}_\mu$ and project on $\vec{n}$ then we get

$$(D_\mu v-\overline{D_\mu v})\vec{n}^\mu\cdot\vec{n}_\nu-(D_\nu v-\overline{D_\nu v})\,\vec{n}_\mu^2-i(D^\mu v+\overline{D^\mu v})H_{\mu\nu}=-i\epsilon^{\mu\nu\lambda\omega}(D_\lambda v+\overline{D_\lambda v})H_{\mu\omega}. \tag{4.12}$$

Both equations lead to the simple expression

$$D_\nu v(u_\mu\bar{u}^\mu)-u_\nu(D_\mu v\bar{u}^\mu)=\epsilon_{\mu\nu\rho\sigma}D^\rho v u^\mu\bar{u}^\sigma \tag{4.13}$$

and its complex conjugate. This expression just constitutes a system of linear homogeneous algebraic equations for the unknowns $D^\mu v$,

$$M_{\mu\nu}D^\mu v=0,\qquad M_{\mu\nu}=(u_\alpha\bar{u}^\alpha)\delta_{\mu\nu}-u_\mu\bar{u}_\nu-\epsilon_{\mu\nu\rho\sigma}u^\rho\bar{u}^\sigma. \tag{4.14}$$

In order to find all solutions of this system of equations we consider the corresponding eigenvalue problem $M_{\mu\nu}D^\mu v=\lambda D_\nu v$. Of course, a solution exists if and only if the determinant vanishes

$$\text{Det}(\hat{M}-\lambda I)=0. \tag{4.15}$$

On the other hand one can find that

$$\text{Det}(\hat{M}-\lambda I)=\lambda(\lambda-u_\mu\bar{u}^\mu)(u_\mu^2\bar{u}_\nu^2-2\lambda u_\mu\bar{u}^\mu+\lambda^2). \tag{4.16}$$

Generically there is a single eigenvalue $\lambda=0$ corresponding to the solution

$$D_\mu v=fu_\mu, \tag{4.17}$$

where f is an arbitrary function. However, if the complex field u obeys the complex eikonal equation $u_\mu^2=0$, then $\lambda=0$ is a degenerate eigenvalue with degeneracy 2. In this case, there exists a second solution. This second solution may be expressed more easily in terms of real vectors. Indeed, if we write $u=a+ib$ then the complex eikonal equation corresponds to

$$a^\mu b_\mu=0,\quad a_\mu^2=b_\mu^2. \tag{4.18}$$

If we introduce analogously $D_\mu v=c_\mu+id_\mu$ then the second solution is given by

$$a^\mu c_\mu=b^\mu c_\mu=a^\mu d_\mu=b^\mu d_\mu=c^\mu d_\mu=0 \tag{4.19}$$

and

$$c_\mu^2 = d_\mu^2. \qquad (4.20)$$

The vector $D_\mu v = c_\mu + i d_\mu$ is unique up to a multiplication by an arbitrary complex function, as befits the solution to a complex, homogeneous linear equation. Conditions (4.19), (4.20) imply that the complex vector $D_\mu v$ has to obey

$$u^\mu D_\mu v = \bar{u}^\mu D_\mu v = 0, \quad D^\mu v D_\mu v = 0 \qquad (4.21)$$

in order to be a solution of the second type. We remark that a wide class of explicitly known instanton configurations, like, e.g., the cylindrically symmetric solutions found by Witten [34], belongs to this second case.

A further possibility, $u_\alpha \bar{u}^\alpha = 0$, which would lead to a even higher degeneracy, is physically uninteresting since it leads to the trivial solutions $u = \text{const}$.

Taking into account formula (4.13) and its general solutions discussed above we find three constraints which are satisfied by all self-dual configurations

$$(D_\lambda v u^\lambda)(u^\beta \bar{u}_\beta) - (D_\lambda v \bar{u}^\lambda) u_\beta^2 = 0, \qquad (4.22)$$

$$(D_\lambda v)^2 (u^\beta \bar{u}_\beta) - (D_\lambda v \bar{u}^\lambda)(D^\beta v u_\beta) = 0, \qquad (4.23)$$

$$(D^\nu v \overline{D_\nu v}) u_\mu^2 - (\overline{D_\nu v} u^\nu)(D_\mu v u^\mu) = 0. \qquad (4.24)$$

4.3 Conserved currents

Following considerations presented, e.g., in [35]–[37], the family of conserved currents may be constructed in the following form

$$j_\mu^G = i(1 + |u|^2)^2 \left(\bar{\pi}_\mu \frac{\partial G}{\partial u} - \pi_\mu \frac{\partial G}{\partial \bar{u}} \right), \qquad (4.25)$$

where G is an arbitrary real function of the complex field u i.e., $G = G(u, \bar{u})$ and π_μ is the canonical momentum (A.1). The four-divergence reads ($G_u \equiv \partial_u G$ etc.)

$$\partial^\mu j_\mu^G = i(1 + |u|^2)^2 \left[G_u \partial_\mu \bar{\pi}^\mu - G_{\bar{u}} \partial_\mu \pi^\mu + G_{uu} u_\mu \bar{\pi}^\mu + G_{u\bar{u}} \bar{u}_\mu \bar{\pi}^\mu - G_{\bar{u}u} u_\mu \pi^\mu - G_{\bar{u}\bar{u}} \bar{u}_\mu \pi^\mu \right] + \\ + 2i(1 + |u|^2)(u\bar{u}_\mu + \bar{u}u_\mu)(G_u \bar{\pi}^\mu - G_{\bar{u}} \pi^\mu). \qquad (4.26)$$

or

$$\partial^\mu j_\mu^G = i(1 + |u|^2)^2 \left[G_u \left(\partial_\mu \bar{\pi}^\mu + \frac{2u}{1 + |u|^2} \bar{u}_\mu \bar{\pi}^\mu \right) - G_{\bar{u}} \left(\partial_\mu \pi^\mu + \frac{2\bar{u}}{1 + |u|^2} \pi_\mu u^\mu \right) \right. \\ \left. + G_{u\bar{u}}(\bar{u}_\mu \bar{\pi}^\mu - u_\mu \pi^\mu) \right] \\ + i(1 + |u|^2)^2 \left[\left(G_{uu} + \frac{2\bar{u}G_u}{1 + |u|^2} \right) u_\mu \bar{\pi}^\mu - \left(G_{\bar{u}\bar{u}} + \frac{2uG_{\bar{u}}}{1 + |u|^2} \right) \bar{u}_\mu \pi^\mu \right]. \qquad (4.27)$$

Taking into account that $\bar{u}_\mu \bar{\pi}^\mu = u_\mu \pi^\mu$ and the pertinent field equations $(1 + |u|^2)\partial_\mu \pi^\mu + 2\bar{u}\pi_\mu u^\mu = 0$ we get

$$\partial^\mu j_\mu^G = i(1 + |u|^2)^2 \left[\left(G_{uu} + \frac{2\bar{u}G_u}{1 + |u|^2} \right) u_\mu \bar{\pi}^\mu - \left(G_{\bar{u}\bar{u}} + \frac{2uG_{\bar{u}}}{1 + |u|^2} \right) \bar{u}_\mu \pi^\mu \right]. \qquad (4.28)$$

Due to the arbitrariness of the function G the currents are conserved if $u_\mu \bar{\pi}^\mu = 0$ and $\bar{u}_\mu \pi^\mu = 0$. These so-called integrability conditions introduce some new relations between degrees of freedom and, in principle, do not have to be satisfied for all solutions of Yang-Mills theory. However, it turns out that in the self-dual sector both conditions hold identically. To prove it observe that

$$\bar{u}_\mu \pi^\mu = \frac{8}{(1+|u|^2)^2} \left[(D^\nu v \overline{D_\nu v}) \bar{u}_\mu^2 - (D_\nu v \bar{u}^\nu) \overline{D_\mu v} \bar{u}^\mu \right], \tag{4.29}$$

where we have used the antisymmetry of $F_{\mu\nu}$ and $K_\mu \bar{u}^\mu \equiv 0$ (where K_μ is defined in (A.3)). The resulting expression is just the complex conjugate of formula (4.24) and therefore equals zero for all configurations of the self-dual sector.

The charges corresponding to the currents (4.25) are

$$Q^G \equiv \int d^3x \, j_0^G \tag{4.30}$$

and obey the algebra of area-preseving diffeomorphisms on the target space two-sphere spanned by the field u under the Poisson bracket, where the fundamental Poisson bracket is (with $x^0 = y^0$)

$$\{u(\mathbf{x}), \pi(\mathbf{y})\} = \{\bar{u}(\mathbf{x}), \bar{\pi}(\mathbf{y})\} = \delta^3(\mathbf{x} - \mathbf{y}), \tag{4.31}$$

as usual. Explicitly, the algebra of area-preserving diffeomorphisms is

$$\{Q^{G_1}, Q^{G_2}\} = Q^{G_3} \quad , \qquad G_3 = i(1+|u|^2)^2(G_{1,\bar{u}} G_{2,u} - G_{1,u} G_{2,\bar{u}}). \tag{4.32}$$

Finally, let us remark that the currents (4.25) are invariant under the residual U(1) gauge transformations that remain after the partial gauge fixing implied by the CFNS decomposition, see eq. (4.5).

4.4 Trivially conserved currents

Using this method we are able to construct more families of infinitely many conserved quantities in self-dual Yang-Mills theory, which are based on other canonical momenta. They are given by the expressions

$$j_\mu^H = \bar{P}_\mu \frac{\partial H}{\partial v} - P_\mu \frac{\partial H}{\partial \bar{v}}, \tag{4.33}$$

$$j_\mu^{\tilde{G}} = \omega_{\mu\nu} \left(\frac{\partial \tilde{G}}{\partial u} u^\nu + \frac{\partial \tilde{G}}{\partial \bar{u}} \bar{u}^\nu \right), \tag{4.34}$$

$$j_\mu^{\tilde{H}} = \omega_{\mu\nu} \left(\frac{\partial \tilde{H}}{\partial v} D^\nu v + \frac{\partial \tilde{H}}{\partial \bar{v}} \overline{D^\nu v} \right), \tag{4.35}$$

where the function $H = H(u, \bar{u}, v\bar{v})$ while the functions $\tilde{G}, \tilde{H}$ depend on the moduli only $\tilde{G} = \tilde{G}(u\bar{u}, v\bar{v}), \tilde{H} = \tilde{H}(u\bar{u}, v\bar{v})$. However, all these currents are trivially conserved. To see this let us analyze the first family in detail. First of all observe that it may be written as

$$j_\mu^H = H'(\bar{v}\bar{P}_\mu - vP_\mu) \tag{4.36}$$

where the prime denotes the derivative w.r.t. $v\bar{v}$, and P_μ is defined in eq. (A.4) . Using the self-dual equations we find that

$$j_\mu^H = \frac{8H'}{(1+|u|^2)^2}\left(\epsilon_{\alpha\mu\beta\gamma}u^\alpha(\bar{v}D^\beta v + v\overline{D^\beta v})\bar{u}^\gamma\right) = \frac{8H'}{(1+|u|^2)^2}\left(\epsilon_{\alpha\mu\beta\gamma}u^\alpha(\bar{v}v^\beta + v\bar{v}^\beta)\bar{u}^\gamma\right).$$

(4.37)

Therefore, these currents are conserved entirely due to the antisymmetry of the $\epsilon_{\alpha\mu\beta\gamma}$ tensor. Analogously one can check that the two remaining families are trivially conserved, as well.

5. Conclusions

The main achievement of the present paper is the derivation of a new family of infinitely many conserved currents for the self-dual sector of classical SU(2) YM theory. This has been accomplished by a combination of techniques developed in the so-called generalized integrability (generalized zero curvature) formulation with a nonlocal transformation of the original gauge degrees of freedom (CFNS decomposition). This alternative procedure provides currents with rather different properties than the previously known ones.

First of all, all calculations are done in a completely covariant manner. Therefore, the obtained currents are conserved for the self-dual sector of SU(2) YM in space-times in any dimension with a completely arbitrary signature.

Secondly, these new currents have a more standard geometrical origin. They are the Noether currents corresponding to the area preserving diffeomorphisms on the two dimensional target space. Therefore they obey the classical diffeomorphism algebra instead of the Kac-Moody or Virasoro ones. Also, the relation between conservation laws and symmetries is different in our case. Although the currents we found generate area-preserving diffeomorphisms on target space, this does not imply that these diffeomorphisms are symmetries of the SDYM equations. The reason is that the SDYM equations in the CFNS decomposition are not Euler-Lagrange, therefore the Noether theorem does not apply (observe that the canonical momenta are derived from the Lagrangian of the original Yang-Mills system, which gives rise to the full Yang-Mills equations).

Thirdly, the currents derived here are given in an explicit form. This is an advantage in comparison with the currents of Prasad and Papachristou, which are given in a more complicated, iterative way and are, therefore, not so easy to work with.

Finally, let us briefly mention some possible generalizations and further directions of future investigations. On the one hand, the procedure employed here is based on the generalized zero curvature condition of ref. [24], which is not restricted to the SDYM. It has been and will be used to detect further integrable sectors in different field theories. On the other hand, recently other nonlocal decompositions of Yang-Mills theory have been proposed, like, e.g., the spin-charge separation of [38]–[39]. It is an interesting question whether these decompositions allow to detect further conservation laws in SDYM. This problem is under current investigation.

Acknowledgments

A. Canonical four momenta and field equations

Here we calculate the canonical momenta

$$
\pi_\mu = \frac{\partial L}{\partial u^\mu} = 8i\frac{1 - |v|^2}{(1 + |u|^2)^2}F_{\mu\nu}\bar{u}^\nu + 16\frac{(1 - |v|^2)^2}{(1 + |u|^2)^4}K_\mu
$$
$$
+ \frac{8}{(1 + |u|^2)^2}\left[(D^\nu v\overline{D_\nu v})\bar{u}_\mu - (D^\nu v\bar{u}_\nu)\overline{D_\mu v}\right], \tag{A.1}
$$

$$
\bar{\pi}_\mu = \frac{\partial L}{\partial \bar{u}^\mu} = -8i\frac{1 - |v|^2}{(1 + |u|^2)^2}F_{\mu\nu}u^\nu + 16\frac{(1 - |v|^2)^2}{(1 + |u|^2)^4}\bar{K}_\mu
$$
$$
+ \frac{8}{(1 + |u|^2)^2}\left[(D^\nu v\overline{D_\nu v})u_\mu - (\overline{D^\nu v}u_\nu)D_\mu v\right] \tag{A.2}
$$

where

$$
K_\mu = (u_\nu\bar{u}^\nu)\bar{u}_\mu - \bar{u}_\nu^2 u_\mu \tag{A.3}
$$

and

$$
P_\mu = \frac{\partial L}{\partial v^\mu} = \frac{8}{(1 + |u|^2)^2}\left[(u_\nu\bar{u}^\nu)\overline{D_\mu v} - (\overline{D^\nu v}u_\nu)\bar{u}_\mu\right], \tag{A.4}
$$

$$
\bar{P}_\mu = \frac{\partial L}{\partial \bar{v}^\mu} = \frac{8}{(1 + |u|^2)^2}\left[(u_\nu\bar{u}^\nu)D_\mu v - (D^\nu v\bar{u}_\nu)u_\mu\right] \tag{A.5}
$$

and finally

$$
\omega_{\mu\nu} = \frac{\partial L}{\partial(\partial^\mu C^\nu)} = 4\left(F_{\mu\nu} - (1 - |v|^2)H_{\mu\nu}\right). \tag{A.6}
$$

The pertinent equations of motion for the complex u field read

$$
\partial_\mu\pi^\mu = L_u = -16i\bar{u}\frac{1 - |v|^2}{(1 + |u|^2)^3}F^{\mu\nu}u_\mu\bar{u}_\nu - 4\cdot 8\bar{u}\frac{(1 - |v|^2)^2}{(1 + |u|^2)^5}\left[(u_\mu\bar{u}^\mu)^2 - u_\mu^2\bar{u}_\nu^2\right]
$$
$$
- \frac{16\bar{u}}{(1 + |u|^2)^3}\left[(u_\mu\bar{u}^\mu)(D^\nu v\overline{D_\nu v}) - (D^\mu v\bar{u}_\mu)(\overline{D_\nu v}u^\nu)\right] \tag{A.7}
$$

$$
\partial_\mu\bar{\pi}^\mu = L_{\bar{u}} = -16iu\frac{1 - |v|^2}{(1 + |u|^2)^3}F^{\mu\nu}u_\mu\bar{u}_\nu - 4\cdot 8u\frac{(1 - |v|^2)^2}{(1 + |u|^2)^5}\left[(u_\mu\bar{u}^\mu)^2 - u_\mu^2\bar{u}_\nu^2\right]
$$
$$
- \frac{16u}{(1 + |u|^2)^3}\left[(u_\mu\bar{u}_\mu)(D^\nu v\overline{D_\nu v}) - (D^\mu v\bar{u}_\mu)(\overline{D_\nu v}u^\nu)\right], \tag{A.8}
$$

while for the complex v field we get

$$\partial_\mu P^\mu = L_v = \frac{-8i\bar{v}}{(1+|u|^2)^2}F^{\mu\nu}u_\mu\bar{u}_\nu + \frac{2\cdot 8\bar{v}(1+|v|^2)}{(1+|u|^2)^4}\left[(u_\mu\bar{u}^\mu)^2 - u_\mu^2\bar{u}_\nu^2\right] +$$
$$+\frac{-8ie}{(1+|u|^2)^2}\left[(u_\mu\bar{u}^\mu)(C^\nu\overline{D_\nu v}) - (C^\mu\bar{u}_\mu)(\overline{D_\nu v}u^\nu)\right] \qquad (A.9)$$

$$\partial_\mu \bar{P}^\mu = L_{\bar{v}} = \frac{-8iv}{(1+|u|^2)^2}F^{\mu\nu}u_\mu\bar{u}_\nu + \frac{2\cdot 8v(1+|v|^2)}{(1+|u|^2)^4}\left[(u_\mu\bar{u}^\mu)^2 - u_\mu^2\bar{u}_\nu^2\right] +$$
$$+\frac{8ie}{(1+|u|^2)^2}\left[(u_\mu\bar{u}^\mu)(C^\nu D_\nu v) - (C^\mu u_\mu)(D_\nu v u^\nu)\right]. \qquad (A.10)$$

The equation for the Abelian gauge field has the form

$$\partial_\mu \omega^{\mu\nu} = \frac{\partial L}{\partial C^\nu} = \frac{-8ie}{(1+|u|^2)^2}\left\{(u_\mu\bar{u}^\mu)\left[v\overline{D^\nu v} - \bar{v}D^\nu v\right] - v\bar{u}^\nu(\overline{D_\mu v}u^\mu) + \bar{v}u^\nu(D_\mu v\bar{u}^\mu)\right\}. \qquad (A.11)$$

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